A NOTE ON BIHARMONIC CURVES IN SASAKIAN SPACE FORMS

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Abstract. We classify the biharmonic non-Legendre curves in a Sasakian space form for which the angle between the tangent vector field and the characteristic vector field is constant and obtain explicit examples of such curves in $\mathbb{R}^{2n+1}(-3)$.

1. Introduction

In 1964, J. Eells and J.H. Sampson introduced the notion of poly-harmonic maps as a natural generalization of harmonic maps ([8]). Thus, while *harmonic maps* between Riemannian manifolds $\phi : (M, g) \to (N, h)$ are the critical points of the *energy functional* $E(\phi) = \frac{1}{2}\int_M |d\phi|^2\ v_g$, the *biharmonic maps* are the critical points of the *bienergy functional* $E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2\ v_g$.

On the other hand, B.-Y. Chen defined the biharmonic submanifolds in an Euclidean space as those with harmonic mean curvature vector field ([6]). If we apply the characterization formula of biharmonic maps to Riemannian immersions into Euclidean spaces, we recover Chen's notion of biharmonic submanifold.

The Euler-Lagrange equation for the energy functional is $\tau(\phi) = 0$, where $\tau(\phi) = \operatorname{trace}\nabla d\phi$ is the tension field, and the Euler-Lagrange equation for the bienergy functional was derived by G. Y. Jiang in [13]:

$$\begin{aligned} \tau_2(\phi) &= -\Delta\tau(\phi) - \operatorname{trace}\ R^N(d\phi, \tau(\phi))d\phi \\ &= 0. \end{aligned}$$

Since any harmonic map is biharmonic, we are interested in non-harmonic biharmonic maps, which are called *proper-biharmonic*.

There are several classification results and some methods to construct biharmonic submanifolds in space forms ([14], [2]). In a natural way, the next step is the study of biharmonic submanifolds in Sasakian space forms. Thus, J. Inoguchi classified in [12] the proper-biharmonic Legendre curves and Hopf cylinders in a 3-dimensional Sasakian space form $M^3(c)$, and in [9] the explicit parametric equations were obtained. In [7], J.T. Cho, J. Inoguchi and J.-E. Lee studied the biharmonic curves in a 3-dimensional Sasakian space forms and T. Sasahara studied the biharmonic integral surfaces in 5-dimensional Sasakian space forms ([18]). New classification results for biharmonic Legendre curves and examples of proper-biharmonic submanifolds in any dimensional Sasakian space form were obtained in [10].

Recent results on biharmonic submanifolds in spaces of nonconstant sectional curvature were obtained by T. Ichiyama, J. Inoguchi and H. Urakawa in [11], by Y.-L. Ou and Z.-P. Wang in [15], and by W. Zhang in [20].

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Biharmonic submanifolds in pseudo-Euclidean spaces were also studied, and many examples and classification results were obtained (for example, see [1], [6]).

The goals of our paper are to obtain new classification results for biharmonic non-Legendre curves in any dimensional Sasakian space form and to obtain explicit equations for some of such curves in $\mathbb{R}^{2n+1}(-3)$.

For a general account of biharmonic maps see [14] and *The Bibliography of Biharmonic Maps* [19].

Conventions. We work in the C^∞ category, that means manifolds, metrics, connections and maps are smooth. The Lie algebra of the vector fields on M is denoted by $C(TM)$.

2. Preliminaries

A triple (φ, ξ, η) is called a *contact structure* on a manifold N^{2n+1}, where φ is a tensor field of type $(1,1)$ on N, ξ is a vector field and η is an 1-form, if

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \forall X, Y \in C(TN).$$

A Riemannian metric g on N is said to be an associated metric and then $(N, \varphi, \xi, \eta, g)$ is a *contact metric manifold* if

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \quad g(X, \varphi Y) = d\eta(X,Y), \quad \forall X, Y \in C(TN).$$

A contact metric structure (φ, ξ, η, g) is called *normal* if

$$N_\varphi + 2d\eta \otimes \xi = 0,$$

where

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X,Y], \quad \forall X, Y \in C(TN),$$

is the Nijenhuis tensor field of φ.

A contact metric manifold $(N, \varphi, \xi, \eta, g)$ is a *Sasakian manifold* if it is normal or, equivalently, if

$$(\nabla_X \varphi)(Y) = g(X,Y)\xi - \eta(Y)X, \quad \forall X, Y \in C(TN).$$

The *contact distribution* of a Sasakian manifold $(N, \varphi, \xi, \eta, g)$ is defined by $\{X \in TN : \eta(X) = 0\}$, and an integral curve of the contact distribution is called *Legendre curve*.

Let $(N, \varphi, \xi, \eta, g)$ be a Sasakian manifold. The sectional curvature of a 2-plane generated by X and φX, where X is an unit vector orthogonal to ξ, is called *φ-sectional curvature* determined by X. A Sasakian manifold with constant φ-sectional curvature c is called a *Sasakian space form* and it is denoted by $N(c)$.

The curvature tensor field of a Sasakian space form $N(c)$ is given by

$$\begin{aligned} R(X,Y)Z = & \tfrac{c+3}{4}\{g(Z,Y)X - g(Z,X)Y\} + \tfrac{c-1}{4}\{\eta(Z)\eta(X)Y - \\ & -\eta(Z)\eta(Y)X + g(Z,X)\eta(Y)\xi - g(Z,Y)\eta(X)\xi + \\ & +g(Z,\varphi Y)\varphi X - g(Z,\varphi X)\varphi Y + 2g(X,\varphi Y)\varphi Z\}. \end{aligned} \tag{2.1}$$

3. Biharmonic non-Legendre curves in Sasakian space forms

Definition 3.1. Let (N^m, g) be a Riemannian manifold and $\gamma : I \to N$ a curve parametrized by arc length, that is $|\gamma'| = 1$. Then γ is called a *Frenet curve of osculating order* r, $1 \le r \le m$, if there exists orthonormal vector fields $E_1, E_2, ..., E_r$ along γ such that $E_1 = \gamma' = T$ and

$$\nabla_T E_1 = \kappa_1 E_2, \quad \nabla_T E_2 = -\kappa_1 E_1 + \kappa_2 E_3, ..., \nabla_T E_r = -\kappa_{r-1} E_{r-1},$$

where $\kappa_1, ..., \kappa_{r-1}$ are positive functions on I.

A Frenet curve of osculating order 1 is a geodesic; a Frenet curve of osculating order 2 with $\kappa_1 =$ constant is called a *circle*; a Frenet curve of osculating order r, $r \geq 3$, with $\kappa_1, ..., \kappa_{r-1}$ constants is called a *helix of order* r and helix of order 3 is called, simply, helix.

Let $(N^{2n+1}, \varphi, \xi, \eta, g)$ be a Sasakian space form with constant φ-sectional curvature c and $\gamma : I \to N$ a non-Legendre Frenet curve of osculating order r with $\eta(T) = f$, where f is a function defined along γ and $f \neq 0$. Since

$$\begin{aligned}\nabla^3_T T = \ & (-3\kappa_1\kappa_1')E_1 + (\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2)E_2 + (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')E_3 \\ & +\kappa_1\kappa_2\kappa_3 E_4\end{aligned}$$

and

$$\begin{aligned}R(T, \nabla_T T)T = \ & \Big(-\tfrac{(c+3)\kappa_1}{4} + \tfrac{(c-1)\kappa_1}{4}f^2\Big)E_2 - \tfrac{(c-1)}{4}ff'T + \tfrac{(c-1)}{4}f'\xi \\ & -\tfrac{3(c-1)\kappa_1}{4}g(E_2, \varphi T)\varphi T,\end{aligned}$$

we get

(3.1)
$$\begin{aligned}\tau_2(\gamma) = \ & \nabla^3_T T - R(T, \nabla_T T)T \\ = \ & (-3\kappa_1\kappa_1' + \tfrac{(c-1)}{4}ff')E_1 + \Big(\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2 + \tfrac{(c+3)\kappa_1}{4} - \tfrac{(c-1)\kappa_1}{4}f^2\Big)E_2 \\ & +(2\kappa_1'\kappa_2 + \kappa_1\kappa_2')E_3 + \kappa_1\kappa_2\kappa_3 E_4 - \tfrac{(c-1)}{4}f'\xi + \tfrac{3(c-1)\kappa_1}{4}g(E_2, \varphi T)\varphi T.\end{aligned}$$

If $c = 1$ then γ is proper-biharmonic if and only if

$$\begin{cases}\kappa_1 = \text{constant} > 0, \quad \kappa_2 = \text{constant} \\ \kappa_1^2 + \kappa_2^2 = 1 \\ \kappa_2\kappa_3 = 0\end{cases}$$

and we can state the following Theorem.

Theorem 3.2. *If $c = 1$ then γ is proper-biharmonic if and only if either γ is a circle with $\kappa_1 = 1$, or γ is a helix with $\kappa_1^2 + \kappa_2^2 = 1$.*

Now, assume that $c \neq 1$. Then γ is proper-bihrmonic if and only if

$$\begin{cases}\kappa_1 = \text{constant} > 0, \\ \kappa_1^2 + \kappa_2^2 = \frac{c+3}{4} - \frac{c-1}{4}f^2 - \frac{1}{\kappa_1^2}\frac{c-1}{4}(f')^2 + \frac{3(c-1)}{4}(g(E_2, \varphi T))^2 \\ \kappa_2' - \frac{1}{\kappa_1}\frac{c-1}{4}f'\eta(E_3) + \frac{3(c-1)}{4}g(E_2, \varphi T)g(E_3, \varphi T) = 0 \\ \kappa_2\kappa_3 - \frac{1}{\kappa_1}\frac{c-1}{4}f'\eta(E_4) + \frac{3(c-1)}{4}g(E_2, \varphi T)g(E_4, \varphi T) = 0\end{cases} \tag{3.2}$$
,

since, from $\eta(T) = g(T, \xi) = f$ and the first Frenet equation, we get $g(\nabla_T T, \xi) = g(\kappa_1 E_2, \xi) = f'$.

Obviously, the above equations are simpler when $f = \eta(T) = \cos\beta_1$ is a constant, where $\beta_1 \in (0, \pi) \setminus \{\frac{\pi}{2}\}$ is the angle between the tangent vector field T and the characteristic vector field ξ.

In the following, we will study only this special case. We have

Theorem 3.3. *Let $c \neq 1$ and γ a Frenet curve of osculating order r such that $\eta(T) = \cos\beta_1 = \text{constant} \notin \{-1, 0, 1\}$. Then γ is proper-biharmonic if and only if either*

a) γ is a circle with $\varphi T = \pm\sin\beta_1 E_2$ and $\kappa_1^2 = 1 + (c-1)\sin^2\beta_1 > 0$,
or
b) γ is a helix with $\varphi T = \pm\sin\beta_1 E_2$ and $\kappa_1^2 + \kappa_2^2 = 1 + (c-1)\sin^2\beta_1 > 0$,
or
c) γ is a Frenet curve of osculating order r, where $r \geq 4$, with

$$\varphi T = \sin\beta_1\cos\beta_2 E_2 + \sin\beta_1\sin\beta_2 E_4$$

and

$$\begin{cases} \kappa_1 = \text{constant} > 0, \quad \kappa_2 = \text{constant} \\ \\ \kappa_1^2 + \kappa_2^2 = \frac{c+3}{4} - \frac{c-1}{4}\cos^2\beta_1 + \frac{3(c-1)}{4}\sin^2\beta_1\cos^2\beta_2 \\ \\ \kappa_2\kappa_3 = -\frac{3(c-1)}{8}\sin^2\beta_1\sin(2\beta_2) \end{cases},$$

where $\beta_2 \in (0, 2\pi)$ is a constant such that $c+3-(c-1)\cos^2\beta_1+3(c-1)sin^2\beta_1\cos^2\beta_2 > 0$, $3(c-1)\sin(2\beta_2) < 0$ and $n \geq 2$.

Proof. First, we see that $\eta(E_2) = g(E_2, \xi) = \frac{1}{\kappa_1}f' = 0$. Next, assume that $g(E_2, \varphi T) = \alpha$, where α is a function defined along γ. Then, using the second Frenet equation, one obtains

$$\alpha' = g(\nabla_T E_2, \varphi T) + g(E_2, \nabla_T\varphi T) = \kappa_2 g(E_3, \varphi T) + g(E_2, \kappa_1\varphi E_2 + \xi - fT),$$

and, since the second term in the right side vanishes, it follows $\kappa_2 g(E_3, \varphi T) = \alpha'$. Now, if γ is proper-biharmonic, replacing into the third equation of (3.2) we obtain

$$\kappa_2\kappa_2' + \frac{3(c-1)}{4}\alpha\alpha' = 0$$

and then

$$\kappa_2^2 + \frac{3(c-1)}{4}\alpha^2 + \omega_0 = 0,$$

where ω_0 is a constant. The second equation of (3.2) becomes

$$\kappa_1^2 + \kappa_2^2 = \frac{c+3}{4} - \frac{c-1}{4}f^2 - \kappa_2^2 - \omega_0.$$

Hence $\kappa_2 = \text{constant}$ and $\alpha = \text{constant}$. If $\kappa_2 = 0$ then, from the biharmonic equation $\tau_2(\gamma) = 0$, we get $E_2 \parallel \varphi T$ and, since $g(\varphi T, \varphi T) = 1 - f^2 = \sin^2\beta_1$ it follows $\varphi T = \pm\sin\beta_1 E_2$. Hence γ is a circle with

$$\kappa_1^2 = \frac{c+3}{4} - \frac{c-1}{4}\cos^2\beta_1 + \frac{3(c-1)}{4}\sin^2\beta_1 = 1 + (c-1)\sin^2\beta_1.$$

Assume that $\kappa_2 \neq 0$. Then $g(E_3, \varphi T) = 0$ and, if $\kappa_3 = 0$ then γ is a helix with

$$\kappa_1^2 + \kappa_2^2 = \frac{c+3}{4} - \frac{c-1}{4}\cos^2\beta_1 + \frac{3(c-1)}{4}\sin^2\beta_1 = 1 + (c-1)\sin^2\beta_1,$$

since, using again the biharmonic equation, one obtains $E_2 \parallel \varphi T$ in this case too. Next, let γ be a proper-biharmonic Frenet curve of osculating order r with $r \geq 4$. Then $g(E_3, \varphi T) = 0$ and, from the biharmonic equation we have $\varphi T \in \text{span}\{E_2, E_4\}$. Since

$$g(\varphi T, \varphi T) = 1 - f^2 = \sin^2\beta_1$$

it follows

$$\varphi T = \sin\beta_1\cos\beta_2 E_2 + \sin\beta_1\sin\beta_2 E_4,$$

where

$$g(E_2,\varphi T) = \alpha = \sin\beta_1\cos\beta_2 \quad \text{and} \quad g(E_4,\varphi T) = \sin\beta_1\sin\beta_2$$

with $\beta_2 = \text{constant} \in (0,2\pi)$.
Finally, if the dimension of N is equal to 3 we can consider an orthogonal system of vectors $\{E = T - f\xi, \varphi T, \xi\}$ along γ and, since f is a constant, it follows easily that $\nabla_T T \parallel \varphi T$. Hence $E_2 \parallel \varphi T$ in this case.

□

A special role in the biharmonic equation $\tau_2(\gamma) = 0$ is played by $g(E_2,\varphi T)$. In the following, we consider γ to be a Frenet curve of osculating order r, with $\eta(T) = f(s) = \cos\beta(s)$ not necessarily constant, such that $E_2 \perp \varphi T$ or $E_2 \parallel \varphi T$.
Case I: $\mathbf{c \neq 1}$, $\mathbf{E_2 \perp \varphi T}$.
In this case γ is proper-biharmonic if and only if

$$\begin{cases} \kappa_1 = \text{constant} > 0, \\ \kappa_1^2 + \kappa_2^2 = \frac{c+3}{4} - \frac{c-1}{4} f^2 - \frac{1}{\kappa_1^2}\frac{c-1}{4}(f')^2 \\ \kappa_2' - \frac{1}{\kappa_1}\frac{c-1}{4} f'\eta(E_3) = 0 \\ \kappa_2\kappa_3 - \frac{1}{\kappa_1}\frac{c-1}{4} f'\eta(E_4) = 0 \end{cases}, \tag{3.3}$$

From $g(E_2,\xi) = \frac{1}{\kappa_1} f'$ one obtains $g(\nabla_T E_2,\xi) - g(E_2,\varphi T) = \frac{1}{\kappa_1} f''$ and then $\kappa_2\eta(E_3)$ $= \frac{1}{\kappa_1} f'' + \kappa_1 f$. Replacing into the third equation of (3.3) one obtains

$$\kappa_2\kappa_2' - \frac{1}{\kappa_1^2}\frac{c-1}{4}(f'f'' + \kappa_1^2 f f') = 0,$$

and then

$$\kappa_2^2 - \frac{1}{\kappa_1^2}\frac{c-1}{4}((f')^2 + \kappa_1^2 f^2) + \omega_1 = 0,$$

where ω_1 is a constant. Now, from the second equation of (3.3) we have

$$\kappa_1^2 + \kappa_2^2 = \frac{c+3}{4} - \kappa_2^2 - \omega_1.$$

Hence $\kappa_2 = \text{constant}$ and $(f'' + \kappa_1^2 f)f' = 0$.

Now, using the Frenet equations, from $g(E_2,\varphi T) = 0$ one obtains $\kappa_2 g(E_3,\varphi T) = -\frac{1}{\kappa_1} f'$ and then, from $\kappa_2 g(E_3,\xi) = \frac{1}{\kappa_1} f'' + \kappa_1 f$, we get

$$\kappa_2\kappa_3 g(E_4,\xi) = \frac{1}{\kappa_1}(f''' + (\kappa_1^2 + \kappa_2^2) f').$$

Since $\tau_2(\gamma) = 0$ implies $\eta(\tau_2(\gamma)) = 0$ one obtains, after a straightforward computation that $f'f''' = 0$. Using this result and differentiating $(f'' + \kappa_1^2 f)f' = 0$ along γ we have

$$\kappa_1^2 (f')^2 + (f'' + \kappa_1^2 f) f'' = 0.$$

We just obtained that $f = \text{constant}$.
We can state

Theorem 3.4. *Assume that $c \neq 1$, $n \geq 2$ and $\nabla_T T \perp \varphi T$. Then γ is proper-biharmonic if and only if either*
a) γ is a circle with $\eta(T) = \cos\beta_0$ and $\kappa_1^2 = \frac{c+3}{4} - \frac{c-1}{4}\cos^2\beta_0$,
or
b) γ is a helix with $\eta(T) = \cos\beta_0$ and $\kappa_1^2 + \kappa_2^2 = \frac{c+3}{4} - \frac{c-1}{4}\cos^2\beta_0$,
where $\beta_0 \in (0, 2\pi) \setminus \{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ is a constant such that $\frac{c+3}{4} - \frac{c-1}{4}\cos^2\beta_0 > 0$.

Remark 3.5. We note that the biharmonic equation $\tau_2(\gamma) = 0$ for curves γ with $\nabla_T T \perp \varphi T$ is equivalent to

$$\Delta H = \frac{1}{4}(c + 3 - (c-1)\cos^2\beta_0)H,$$

i.e. H is an eigenvector of Δ, where $H = \nabla_T T$ is the mean curvature vector field of γ.

Case II: $\mathbf{c \neq 1}$, $\mathbf{E_2 \parallel \varphi T}$.
In this case $g(E_2, \xi) = \frac{1}{\kappa_1} f' = 0$ and then $f = \cos\beta_0 =$ constant. Since $g(\varphi T, \varphi T) = 1 - (g(T,T))^2 = \sin^2\beta_0$ we have $\varphi T = \pm\sin\beta_0 E_2$.
We obtain

Proposition 3.6. *Assume that $c \neq 1$ and $\nabla_T T \parallel \varphi T$. Then γ is proper-biharmonic if and only if either*
a) γ is a circle with $\eta(T) = \cos\beta_0$ and $\kappa_1^2 = c - (c-1)\cos^2\beta_0$,
or
b) γ is a helix with $\eta(T) = \cos\beta_0$ and $\kappa_1^2 + \kappa_2^2 = c - (c-1)\cos^2\beta_0$,
where $\beta_0 \in (0, 2\pi) \setminus \{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ is a constant such that $c - (c-1)\cos^2\beta_0 > 0$.

Next, let γ be a proper-biharmonic non-Legendre curve with $\nabla_T T \parallel \varphi T$. As $\varphi T = \pm\sin\beta_0 E_2$ one obtains after a straightforward computation that

$$\nabla_T E_2 = -\frac{1}{\sin\beta_0}\Big(\frac{\kappa_1}{\sin\beta_0} \pm \cos\beta_0\Big)T + \frac{1}{\sin\beta_0}\Big(\frac{\kappa_1\cos\beta_0}{\sin\beta_0} \pm 1\Big)\xi.$$

Using the second Frenet equation we have

$$\kappa_2^2 = \frac{(\kappa_1\cos\beta_0 \pm \sin\beta_0)^2}{\sin^2\beta_0}.$$

Thus γ is a circle if and only if $\kappa_1 = \mp\tan\beta_0 > 0$. From Proposition 3.6 we easily get that γ is a proper-biharmonic circle if and only if

$$\kappa_1^2 = \frac{c - 1 + \sqrt{c^2 - 2c + 5}}{2} \quad \text{and} \quad \cos^2\beta_0 = \frac{c + 1 - \sqrt{c^2 - 2c + 5}}{2(c-1)}.$$

If $\kappa_2 \neq 0$, from the expression of κ_2 and the third Frenet equation it follows that $\kappa_3 = 0$. Hence γ is a helix. Now, γ is proper-biharmonic if and only if κ_1 satisfies

$$\kappa_1^2 \pm \cos(2\beta_0)\kappa_1 + (1-c)\sin^4\beta_0 = 0$$

and $\beta_0 \in (0, 2\pi) \setminus \{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ if $c > 1$ or $\beta_0 \in (0, 2\pi) \setminus \{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ such that $\cos\beta_0 \in \Big(-\sqrt{\frac{c-1}{c-2}}, \sqrt{\frac{c-1}{c-2}}\Big)$ if $c < 1$.
We conclude with the following

Theorem 3.7. *If $c \neq 1$ and $\nabla_T T \parallel \varphi T$, then γ is a Frenet curve of osculating order $r \leq 3$ and it is proper-biharmonic if and only if either*
a) γ is a circle with $\eta(T) = \pm\sqrt{\frac{c+1-\sqrt{c^2-2c+5}}{2(c-1)}}$ and $\kappa_1^2 = \frac{c-1+\sqrt{c^2-2c+5}}{2}$,
or

b) γ is a helix with $\eta(T)=\cos\beta_0$ and κ_1 satisfies

$$\kappa_1^2 \pm \cos(2\beta_0)\kappa_1 + (1-c)\sin^4\beta_0 = 0,$$

where $\beta_0 = \text{constant} \in (0,2\pi)\setminus\{\frac{\pi}{2},\pi,\frac{3\pi}{2}\}$ if $c>1$ or $\beta_0 = \text{constant} \in (0,2\pi)\setminus\{\frac{\pi}{2},\pi,\frac{3\pi}{2}\}$ such that $\cos\beta_0 \in \left(-\sqrt{\frac{c-1}{c-2}},\sqrt{\frac{c-1}{c-2}}\right)$ if $c<1$. In the last case $\kappa_2^2 = (\kappa_1\cot\beta_0 \pm 1)^2$.

Remark 3.8. A curve γ with $\nabla_T T \parallel \varphi T$ is proper-biharmonic if and only if

$$\Delta H = (c-(c-1)\cos^2\beta_0)H,$$

where H is the mean curvature vector field of γ.

4. Biharmonic curves in $\mathbb{R}^{2n+1}(-3)$

While proper-biharmonic Legendre curves exist only in a Sasakian space form $N^{2n+1}(c)$ with constant φ-sectional curvature c bigger than 1 if $n=1$, or -3 if $n>1$ (see [12], [10]), proper-biharmonic non-Legendre curves can be found in Sasakian space forms with any φ-sectional curvature.

We mention that, in the case when $c=-3$, T. Sasahara studied in [16] the submanifolds in the Sasakian space form $\mathbb{R}^{2n+1}(-3)$ whose φ-mean curvature vectors are eigenvectors of the Laplacian and in [17] the Legendre surfaces in $\mathbb{R}^5(-3)$ for which mean curvature vectors field are eigenvectors of the Laplacian.

In this section we obtain the explicit equations for proper-biharmonic circles with $E_2 \perp \varphi T$ and for all proper-biharmonic curves with $E_2 \parallel \varphi T$ in $\mathbb{R}^{2n+1}(-3)$.

First, let us recall briefly some notions and results about the structure of the Sasakian space form $\mathbb{R}^{2n+1}(-3)$ as they are presented in [3].

Consider on $\mathbb{R}^{2n+1}(-3)$, with elements of the form $(x^1,...,x^n,y^1,...,y^n,z)$, its standard contact structure defined by the 1-form $\eta = \frac{1}{2}(dz - \sum_{i=1}^n y^i dx^i)$, the characteristic vector field $\xi = 2\frac{\partial}{\partial z}$ and the tensor field φ given by the matrix

$$\begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y^j & 0 \end{pmatrix}.$$

Then $g = \eta\otimes\eta + \frac{1}{4}\sum_{i=1}^n((dx^i)^2+(dy^i)^2)$ is an associated Riemannian metric and $(\mathbb{R}^{2n+1},\varphi,\xi,\eta,g)$ is a Sasakian space form with constant φ-sectional curvature equal to -3, denoted $\mathbb{R}^{2n+1}(-3)$.

The vector fields $X_i = 2\frac{\partial}{\partial y^i}$, $X_{n+i} = \varphi X_i = 2(\frac{\partial}{\partial x^i} + y^i\frac{\partial}{\partial z})$, $i=1,...,n$, and $\xi = 2\frac{\partial}{\partial z}$ form an orthonormal basis in $\mathbb{R}^{2n+1}(-3)$ and after straightforward computations one obtains

$$[X_i,X_j] = [X_{n+i},X_{n+j}] = [X_i,\xi] = [X_{n+i},\xi] = 0, \quad [X_i,X_{n+j}] = 2\delta_{ij}\xi$$

and

$$\nabla_{X_i}X_j = \nabla_{X_{n+i}}X_{n+j} = 0, \quad \nabla_{X_i}X_{n+j} = \delta_{ij}\xi, \quad \nabla_{X_{n+i}}X_j = -\delta_{ij}\xi,$$
$$\nabla_{X_i}\xi = \nabla_\xi X_i = -X_{n+i}, \quad \nabla_{X_{n+i}}\xi = \nabla_\xi X_{n+i} = X_i$$

for any $i,j=1,...,n$.

Now, let $\gamma : I \to \mathbb{R}^{2n+1}(-3)$ be a Frenet curve of osculating order $r>1$, parametrized by arc length, with the tangent vector field $T=\gamma'$ given by

$$T = \sum_{i=1}^n (T_i X_i + T_{n+i}X_{n+i}) + \cos\beta_0\xi, \tag{4.1}$$

where $\cos\beta_0$ is a constant. Using the above formulas for the Levi-Civita connection we have

$$\nabla_T T=\sum_{i=1}^{n}((T_i'+2\cos\beta_0 T_{n+i})X_i+(T_{n+i}'-2\cos\beta_0 T_i)X_{n+i}) \tag{4.2}$$

From Theorems 3.4 and 3.7, using the same techniques as in [4], [5] and [7], we get

Theorem 4.1. *The parametric equations of proper-biharmonic circles parametrized by arc length in $\mathbb{R}^{2n+1}(-3)$, $n\geq 2$, with $\nabla_T T\perp\varphi T$, are*

$$\left\{\begin{array}{rcl} x^i(s) & = & \pm\frac{1}{\kappa_1}(2\sin(\kappa_1 s)c_1^i\mp 2\cos(\kappa_1 s)c_2^i-\cos(2\kappa_1 s)d_1^i\\ & & -\sin(2\kappa_1 s)d_2^i)+a^i\\ y^i(s) & = & \frac{1}{\kappa_1}(2\cos(\kappa_1 s)c_1^i\pm 2\sin(\kappa_1 s)c_2^i+\sin(2\kappa_1 s)d_1^i\\ & & -\cos(2\kappa_1 s)d_2^i)+b^i\\ z(s) & = & \pm\frac{2}{\kappa_1}(1+\sum_{i=1}^{n}((c_1^i)^2+(c_2^i)^2))s\\ & & +\frac{1}{2\kappa_1^2}\sum_{i=1}^{n}(\pm\cos(4\kappa_1 s)d_1^i d_2^i-2\cos(2\kappa_1 s)c_1^i c_2^i\\ & & +4\cos(3\kappa_1 s)c_2^i d_2^i-4\sin(3\kappa_1 s)c_1^i d_2^i)\\ & & \mp\frac{1}{\kappa_1}\sum_{i=1}^{n}b^i(-2\sin(\kappa_1 s)c_1^i\pm 2\cos(\kappa_1 s)c_2^i\\ & & +\cos(2\kappa_1 s)d_1^i+\sin(2\kappa_1 s)d_2^i)+e \end{array}\right. , \tag{4.3}$$

where $\kappa_1^2=\cos^2\beta_0$, $\beta_0\in(0,2\pi)\setminus\{\frac{\pi}{2},\pi,\frac{3\pi}{2}\}$ is a constant, and a^i, b^i, c_1^i, c_2^i, d_1^i, d_2^i and e are constants such that the n-dimensional constant vectors $c_j=(c_j^1,...,c_j^n)$ and $d_j=(d_j^1,...,d_j^n)$, $j=1,2$, satisfy

$$\left\{\begin{array}{l} |c_1|^2+|c_2|^2+|d_1|^2+|d_2|^2=\sin^2\beta_0\\ \langle c_1,d_1\rangle\pm\langle c_2,d_2\rangle=0,\ \ \langle c_1,d_2\rangle\mp\langle c_2,d_1\rangle=0 \end{array}\right. .$$

Proof. Let $\gamma:I\to\mathbb{R}^{2n+1}(-3)$ be a circle parametrized by arc length, with the tangent vector field $T=\gamma'$ given by (4.1) and $\nabla_T T\perp\varphi T$. From the equation (4.2) one obtains

$$E_2=\frac{1}{\kappa_1}\sum_{i=1}^{n}((T_i'+2\cos\beta_0 T_{n+i})X_i+(T_{n+i}'-2\cos\beta_0 T_i)X_{n+i})$$

and, using $g(E_2,\varphi T)=0$, a direct computation shows that

$$\begin{array}{rcl}\nabla_T E_2 & = & \frac{1}{\kappa_1}(\sum_{i=1}^{n}((T_i'+2\cos\beta_0 T_{n+i})'+(T_{n+i}'-2\cos\beta_0 T_i)\cos\beta_0)X_i\\ & & +((T_{n+i}'-2\cos\beta_0 T_i)'-(T_i'+2\cos\beta_0 T_{n+i})\cos\beta_0)X_{n+i})\end{array}$$

and, since γ is a circle, it follows

$$\left\{\begin{array}{l} A_i'+B_i\cos\beta_0=0\\ B_i'-A_i\cos\beta_0=0 \end{array}\right. , \tag{4.4}$$

where $A_i = \frac{1}{\kappa_1}(T_i' + 2\cos\beta_0 T_{n+i})$ and $B_i = \frac{1}{\kappa_1}(T_{n+i}' - 2\cos\beta_0 T_i)$.
Solving (4.4) and imposing for γ to be proper-biharmonic, according to Theorem 3.4 that is $\kappa_1 = \pm\cos\beta_0 > 0$, we get the following equations

$$\begin{cases} T_i' \pm 2\kappa_1 T_{n+i} = \kappa_1 \cos(\kappa_1 s) c_1^i \pm \kappa_1 \sin(\kappa_1 s) c_2^i \\ T_{n+i}' \mp 2\kappa_1 T_i = \pm\kappa_1 \sin(\kappa_1 s) c_1^i - \kappa_1 \cos(\kappa_1 s) c_2^i \end{cases},$$

which general solutions are

$$\begin{cases} T_i = -\sin(\kappa_1 s) c_1^i \pm \cos(\kappa_1 s) c_2^i + \cos(2\kappa_1 s) d_1^i + \sin(2\kappa_1 s) d_2^i \\ T_{n+i} = \pm\cos(\kappa_1 s) c_1^i + \sin(\kappa_1 s) c_2^i \pm \sin(2\kappa_1 s) d_1^i \mp \cos(2\kappa_1 s) d_2^i \end{cases},$$

where c_1^i, c_2^i, d_1^i and d_2^i are constants, such that

$$\begin{cases} \sum_{i=1}^n ((c_1^i)^2 + (c_2^i)^2 + (d_1^i)^2 + (d_2^i)^2) = \sin^2\beta_0 \\ \sum_{i=1}^n ((c_1^i)(d_1^i) \pm (c_2^i)(d_2^i)) = 0, \quad \sum_{i=1}^n ((c_1^i)(d_2^i) \mp (c_2^i)(d_1^i)) = 0 \end{cases},$$

since $g(T,T) = 1$.
Finally, replacing into expression of γ' and integrating we get (4.3). □

Remark 4.2. In order to find explicit examples of proper-biharmonic curves with $\nabla_T T \perp \varphi T$ in $\mathbb{R}^{2n+1}(-3)$ we will stick at proper-biharmonic circles since the computations in the case of helices are rather complicated.

Theorem 4.3. *Proper-biharmonic curves in $\mathbb{R}^{2n+1}(-3)$, with $\nabla_T T \parallel \varphi T$, are either*
a) Proper-biharmonic circles given by

$$(4.5)\quad \begin{cases} x^i(s) = & (\sqrt{5}+1)\Big(\cos\left(\frac{\sqrt{5}-1}{2}s\right)c_1^i + \sin\left(\frac{\sqrt{5}-1}{2}s\right)c_2^i\Big) + a^i \\ y^i(s) = & (\sqrt{5}+1)\Big(\sin\left(\frac{\sqrt{5}-1}{2}s\right)c_1^i - \cos\left(\frac{\sqrt{5}-1}{2}s\right)c_2^i\Big) + b^i \\ z(s) = & \frac{1-\sqrt{5}\pm 2\sqrt{1+\sqrt{5}}}{2}s + \frac{3+\sqrt{5}}{2}\sum_{i=1}^n(((c_1^i)^2 - (c_2^i)^2)\sin((\sqrt{5}-1)s) \\ & -2\cos((\sqrt{5}-1)s)c_1^i c_2^i) + (1+\sqrt{5})\sum_{i=1}^n b_i\Big(\sin\left(\frac{\sqrt{5}-1}{2}s\right)c_2^i \\ & +\cos\left(\frac{\sqrt{5}-1}{2}s\right)c_1^i\Big) + d \end{cases},$$

where a^i, b^i, c_1^i, c_2^i and d are constants such that the n-dimensional constant vectors $c_j = (c_j^1, ..., c_j^n)$, $j = 1, 2$, satisfy

$$|c_1|^2 + |c_2|^2 = \frac{3-\sqrt{5}}{4}.$$

or

b) Proper-biharmonic helices given by

$$
(4.6)\quad \begin{cases}
x^i(s) = & -\frac{2\kappa_1}{\kappa_1\pm\sin(2\beta_0)}\Big(\cos\Big(\frac{\kappa_1\pm\sin(2\beta_0)}{\kappa_1}s\Big)c_1^i+\sin\Big(\frac{\kappa_1\pm\sin(2\beta_0)}{\kappa_1}s\Big)c_2^i\Big)+a^i\\
y^i(s) = & \frac{2\kappa_1}{\kappa_1\pm\sin(2\beta_0)}\Big(\sin\Big(\frac{\kappa_1\pm\sin(2\beta_0)}{\kappa_1}s\Big)c_1^i-\cos\Big(\frac{\kappa_1\pm\sin(2\beta_0)}{\kappa_1}s\Big)c_2^i\Big)+b^i\\
z(s) = & 2\Big(\cos\beta_0+\frac{\kappa_1\sin^2\beta_0}{\kappa_1\pm\sin(2\beta_0)}\Big)s+\frac{\kappa_1^2}{(\kappa_1\pm\sin(2\beta_0))^2}\\
& \cdot\Big(\sin\Big(\frac{2(\kappa_1\pm\sin(2\beta_0))}{\kappa_1}s\Big)\sum_{i=1}^n((c_1^i)^2-(c_2^i)^2)\\
& +\cos\Big(\frac{2(\kappa_1\pm\sin(2\beta_0))}{\kappa_1}s\Big)\sum_{i=1}^n(c_1^ic_2^i)\Big)\\
& -\frac{2\kappa_1}{\kappa_1\pm\sin(2\beta_0)}\sum_{i=1}^n b^i\Big(\cos\Big(\frac{\kappa_1\pm\sin(2\beta_0)}{\kappa_1}s\Big)c_1^i\\
& +\sin\Big(\frac{\kappa_1\pm\sin(2\beta_0)}{\kappa_1}s\Big)c_2^i\Big)+d
\end{cases},
$$

where $\beta_0\in(0,2\pi)\backslash\{\frac{\pi}{2},\pi,\frac{3\pi}{2}\}$ *is a constant such that* $\cos\beta_0\in\Big(-1,-\frac{2\sqrt{5}}{5}\Big)\cup\Big(\frac{2\sqrt{5}}{5},1\Big)$, κ_1 *is a positive solution of the equation*

$$\kappa_1^2\pm\sin(2\beta_0)\kappa_1+4\sin^4\beta_0=0$$

and a^i, b^i, c_1^i, c_2^i *and* d *are constants such that*

$$|c_1|^2+|c_2|^2=\sin^2\beta_0.$$

Proof. We will prove only the first statement because the second one can be obtained in a similar way by the meaning of Theorem 3.7.
Assume that γ is a proper-biharmonic circle in $\mathbb{R}^{2n+1}(-3)$ parametrized by arc length, such that $\nabla_T T\parallel\varphi T$. Then, from (4.2) and since $\varphi T=\sum_{i=1}^n(-T_{n+i}X_i+T_iX_{n+i})$, $g(\varphi T,\varphi T)=\sin^2\beta_0$, where $\eta(T)=\cos\beta_0$, one obtains

$$T_i'=\Big(\mp\frac{\sin(2\beta_0)}{\kappa_1}-1\Big)T_{n+i},\qquad T_{n+i}'=\Big(\pm\frac{\sin(2\beta_0)}{\kappa_1}+1\Big)T_i$$

Now, since γ is a proper-biharmonic circle we get, from Theorem 3.7, $\kappa_1=\mp\tan\beta_0>0$ and $\cos^2\beta_0=\frac{1+\sqrt{5}}{4}$ and hence the above equations become

$$T_i'=\frac{\sqrt{5}-1}{2}T_{n+i},\qquad T_{n+i}'=\frac{1-\sqrt{5}}{2}T_i,$$

with general solutions

$$T_i=\cos\Big(\frac{\sqrt{5}-1}{2}\Big)c_1^i+\sin\Big(\frac{\sqrt{5}-1}{2}\Big)c_2^i,\quad T_{n+i}=\cos\Big(\frac{\sqrt{5}-1}{2}\Big)c_2^i-\sin\Big(\frac{\sqrt{5}-1}{2}\Big)c_1^i,$$

where c_1^i and c_2^i, $i=1,...,n$, are constants.
Replacing in the expression of $T=\gamma'$, integrating and imposing $g(T,T)=1$ we obtain the conclusion. □

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